Journal of Applied Mathematics and Mechanics 72 (2008) 33–35

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The properties of asymptotic expansions for the parameters of unsteady transonic flows with axial symmetry☆

A.N. Bogdanov, V.N. Diesperov

Moscow, Russia

ARTICLE INFO

Article history:
Received 20 December 2005

ABSTRACT

A modified equation, which refines the model of unsteady transonic flow with axial symmetry, is proposed, and the following problems are considered: the determination of the wave front when a weak perturbation propagates and the formation of a shock wave from a perturbation with a continuous profile.

Within the framework of the theory of small perturbations, unsteady transonic flows are described by the classical Lin–Reissner–Tsien (LRT) equation.[1] However, if, according to the physical meaning of the problem being considered, it is required that the perturbations in the flow should propagate at a finite velocity in all directions (as in flows described on the basis of the complete Euler equations), then, instead of the LRT equation, it is necessary to use a modified equation which has a singular term with the second derivative of the perturbed velocity potential with respect to time.[2] This term arises naturally when deriving the asymptotic equation of unsteady transonic flow from the complete equations for the velocity potential. Investigation of plane transonic flows using the refined model (taking account of the singular term) has enabled many unsteady and non-linear phenomena to be described more correctly.[3–5] Unsteady transonic flow with axial symmetry is investigated below using the usual asymptotic model and the refined model. It is shown that regularization of the model is necessary in this case in order to obtain the complete flow pattern. The results obtained are qualitatively identical to those obtained earlier for plane symmetric flow, but the quantitative differences can be significant (close to the axis of symmetry of the problem).

1. Characteristic fronts in a transonic axisymmetric flow

We shall assume that unsteady transonic flow in a cylindrical system of coordinates x, r is a perturbation of a certain constant one-dimensional flow in the direction of the x axis (which is the axis of symmetry of the problem). By analogy with plane symmetric flows,[2] we will write the modified Lin–Reissner–Tsien (LRT) equation in the case of axisymmetric flow. For convenience in comparing the axisymmetric and plane cases, we introduce the parameter ν such that $\nu = 0$ for the plane case and $\nu = 1/2$ for the case of axial symmetry. For the velocity potential, we have

$$\varepsilon\phi_{tt} + \phi_{tx} + (K_\infty - (\gamma+1)\phi_t)\phi_{xx} - \phi_{rr} - 2\nu r^{-1}\phi_r = 0;$$
$$\varepsilon \ll 1, \quad K_\infty = (M_\infty^2 - 1)/\varepsilon \tag{1.1}$$

where γ is the ratio of the specific heat capacities and M_∞ is the Mach number of the unperturbed flow. When $\varepsilon = 0$ and $\nu = 0$, the well-known LRT equation is obtained from Eq. (1.1). The LRT equation is the fundamental equation of unsteady transonic flow in the asymptotic transonic model (which is also the basis for calling Eq. (1.1) the modified LRT equation).

The characteristics surfaces can be obtained in the case of Eq. (1.1) by following a well-known approach;[6] they are cones $\xi = \xi(t, x, r)$ and satisfy the equation

$$K^*\xi_x^2 + \xi_r^2 + 2\nu r^{-1}\xi_r - \xi_t\xi_x - \varepsilon\xi_t^2 = 0;$$
$$K^* = K_\infty - (\gamma+1)\phi_x \tag{1.2}$$

The equation of the plane tangential to the characteristic surfaces at the point (x_0, r_0) can be written in the form

$$\xi = t + \alpha(x - x_0) + \beta(r - r_0) = 0 \tag{1.3}$$

Eliminating ξ from the equalities (1.2) and (1.3) we obtain the condition relating the coefficients α and β, whence it follows that

$$\beta = \nu r^{-1} \pm R_1, \quad R_1^2 = \nu^2 r^{-2} - K^*\alpha^2 + 2\alpha + \varepsilon \tag{1.4}$$

Hence, the family of planes (1.3) with condition (1.4) is a single-parameter family. The characteristic cone is the envelope of planes (1.3) when α varies. The existence of just a single parameter in the equation of the plane leads to the fact that the planes constituting the family are transformed into one another by rotation about the

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 1, pp. 54–57, 2008.

E-mail address: bogdanov@imec.msu.ru (A.N. Bogdanov).

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2008.03.009

axis which is orthogonal to the plane t = const (that is, this axis is the time axis), which would be expected, since the intersection of the characteristic cone by the plane t = const is the characteristic at the given instant of time. The non-linearity in Eq. (1.1), associated with the coefficients K^*, only leads to a change in the propagation velocity of the characteristic surface (and some fronts can overtake others, see below in Section 2) which, nevertheless, qualitatively remains a surface of the same type as it was when K^* = const.

The equation of the envelope of the family of planes (1.3), with coefficients satisfying condition (1.4), is given by the system of equations

$$\xi = t + \alpha(x - x_0) + [\nu r^{-1} \pm R_1](r - r_0) = 0$$
$$\xi_\alpha = 0 \tag{1.5}$$

Solving the equation

$$\xi_\alpha = (x - x_0) \pm (r - r_0)(1 - K^*\alpha)R_1^{-1} = 0$$

for α, we obtain

$$\alpha = \frac{1}{K^*}(1 \pm (x - x_0)R_2), \quad R_2^2 = \frac{1 + (\nu^2 r^{-2} + \varepsilon)K^*}{(x - x_0)^2 + K^*(r - r_0)^2} \tag{1.6}$$

Then, by Eq. (1.4),

$$\beta = \nu r^{-1} \pm (r - r_0)R_2 \tag{1.7}$$

We choose the upper sign in expressions (1.6) and (1.7), which corresponds to the propagation of a perturbation from the source. The equation of the characteristic surface can now be obtained and, eliminating the parameter α in system (1.5) using equalities (1.6), we have

$$K^*t^2 + 2t(x - x_0) - (\nu^2 r^{-2} + \varepsilon)(x - x_0)^2 + 2\nu r^{-1}$$
$$(K^*t + (x - x_0))(r - r_0) =$$
$$= (r - r_0)^2(1 + \varepsilon K^*) \tag{1.8}$$

When K^* = const, Eq. (1.8) for each fixed t describes ellipsoids in the cylindrical system of coordinates x, r. If $\nu = 0$ (that is, x and r are Cartesian coordinates) Eq. (1.8) becomes the equation of the characteristic front in the plane case.[2]

When $\varepsilon = 0$ (also supposing $x_0 = r_0 = 0$), Eq. (1.8) becomes an equation of the form

$$K^*t^2 + 2tx - \nu^2 r^{-2} x^2 + 2\nu(K^*t + x) = r^2 \tag{1.9}$$

In the plane case ($\nu = 0$ and, then, x and r are Cartesian coordinates), only the first two terms remain on the left-hand side of Eq. (1.9) and its solutions for a specified value of t are parabolae in the x, r plane. These solutions give infinite propagation velocities of the wave fronts of weak perturbations downstream and are unreal from a physical point of view. Note that, according to Eq. (1.2) when $\varepsilon = 0$, the planes t = const will also be characteristic surfaces. The formal corollary of this fact is infinite propagation velocities of perturbations downstream and the fact that of the Cauchy problem for the LRT equation is ill-posed.

In the case of axial symmetry ($\nu = 1/2$), the degeneration, when $\varepsilon = 0$, of the characteristic front from a point source into an unclosed curve, which is observed in the plane case,[2] also occurs, and Eq. (1.9), for a fixed t, describes an elliptic paraboloid.

On putting $K^* = 0$ for simplicity, we obtain the following two solutions of Eq. (1.9)

$$r = \pm\sqrt{x[t + \nu + \sqrt{(t + \nu)^2 - \nu^2}]} \tag{1.10}$$

corresponding to the characteristic fronts. In the plane case when $\nu = 0$, they correspond to parabolae. When $\nu = 0$, the two further solutions

$$r = \pm\sqrt{x[t + \nu - \sqrt{(t + \nu)^2 - \nu^2}]} \tag{1.11}$$

become straight lines $r = 0$. It is seen that, after a time t, the characteristic front (1.10) departs from the axis of symmetry x to a greater distance than in the plane case, and the characteristic front (1.11) to a shorter distance. In this case, as time passes, the fronts (1.10) asymptotically approach the characteristic fronts in the plane case and the characteristic fronts (1.11) asymptotically approach $r = 0$. Obviously, solutions (1.10) correspond to the real flow pattern, and the difference between the plane and axisymmetric cases are only substantial in the case of short times. Solution (1.11) can be eliminated if the term $\nu^2 r^{-2} x^2$ in Eq. (1.9) is neglected, which is natural in the case of the propagation of the characteristic front from the source to large values of r. Obviously, it also follows that Eq. (1.8) is treated in the same way.

Hence, in the case of axial symmetry of unsteady transonic flow, as in the plane case, it is necessary to take account of the singular terms of the transonic expansion to regularize the transient problem.

2. The formation of shock waves

We will now consider the occurrence, as time passes, of the intersection of the characteristics (1.8) (the so-called reversal of the wave front of a perturbation) which implies the formation of a shock wave.

We choose two sources of perturbations with the coordinates (x_{01}, r_0) and (x_{02}, r_0). Using Eq. (1.8), we calculate $\partial x/\partial x_0$ along the ray $r = r_0$ (where $\partial r/\partial x_0 = 0$, $\partial r_0/\partial x_0 = 0$). The condition for the characteristics to intersect then has the form

$$\frac{\partial x}{\partial x_0} = 1 - \frac{A}{2}\frac{\partial K^*}{\partial x_0} = 0; \quad A = \frac{t^2}{t - (\nu^2 r^{-2} + \varepsilon)(x - x_0)}$$

Taking account of the fact that

$$\frac{\partial K^*}{\partial x_0} = -(\gamma + 1)\frac{\partial u}{\partial x_0}$$

we find

$$\frac{\partial u}{\partial x_0} = -\frac{2}{(\gamma + 1)A} \tag{2.1}$$

When $\varepsilon = 0$, we have $\partial u/\partial x_0 < 0$. This means that an envelope only arises in the case of compressive perturbations and it occurs earlier (for shorter t), the greater the absolute magnitude of the velocity gradient. When $\varepsilon = 0$, the right-hand side of Eq. (2.1) falls to zero with time, that is, even perturbations which are initially very weak reverse after some sufficiently long time.

Relation (2.1) shows that, in the case when $\varepsilon \neq 0$, intersection of the characteristics only begins in compressive perturbations and starts later (for the same initial amplitude of the perturbation) in perturbations which propagate downstream ($x > x_0$). In analysing relation (2.1), it should be remembered that, when $\varepsilon = 0$, the model does not totally describe perturbations (or segments of their fronts) which propagate downstream (when $x > x_0$).

Using relation (2.1), the time necessary for the characteristics to intersect can be calculated for a specified initial profile of the perturbation wave $\partial u/\partial x_0$. Solving Eq. (2.1) for t, we obtain two solutions. In the case of perturbations travelling downstream ($x > x_0$), the time of intersection of the characteristics (we recall that intersection of the characteristics only occurs when $\partial u/\partial x_0 < 0$) is as follows:

$$t_* = -\left((\gamma+1)\frac{\partial u}{\partial x_0}\right)^{-1}(1+R_3);$$

$$R_3^2 = 1 + 4(\nu^2 r^{-2} + \varepsilon)(x - x_0)$$

and the second root gives a negative time and must be discarded as having no physical meaning. It is seen that, in the case of an axisymmetric flow ($\nu = 1/2$), the magnitude of t_* increases substantially as r decreases. In the case of perturbations travelling upstream ($x < x_0$), we have two positive values of the time of intersection of the characteristics:

$$t_*^{\pm} = -\left((\gamma+1)\frac{\partial u}{\partial x_0}\right)^{-1}(1 \pm R_3)$$

Obviously, the intersection of the characteristics occurs after a shorter time, that is, after a time t_*^{-}, which has physically been confirmed. In the case of axisymmetric flow ($\nu = 1/2$), this time is reduced and the more so, the smaller the value of r.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (007-01-00589) and the Program for the Support of Leading Scientific Schools (NSh-6791.2006.1).

References

1. Lin CC, Reissner E, Tsien HS. On two-dimensional non-steady motion of a slender body in a compressible fluid. *J Math and Phys* 1948;**27**(3):220–31.
2. Bogdanov AN. Higher approximations of the transonic expansion in problems of unsteady transonic flow. *Prikl Mat Mekh* 1997;**61**(5):798–811.
3. Bogdanov AN. Simulation of the transition mode of operation of a transonic nozzle. *Mat Modelirovaniye* 1995;**7**(9):117–26.
4. Bogdanov AN, Diesperov VN. The simulation of unsteady transonic flow and the stability of a transonic boundary layer. *Prikl Mat Mekh* 2005;**69**(3):394–403.
5. Bogdanov AN, Diesperov VN. Tollmien-Schlichting waves in a transonic boundary layer. Excitation from the outer flow and from the surface. *Prikl Mat Mekh* 2007;**71**(2):289–300.
6. Cole JD, Cook LP. *Transonic aerodynamics*. Amsterdam etc.: North–Holland; 1986.

Translated by E.L.S.